Theory of Gyrotron Traveling-Wave Amplifiers

Q. F. LI, S. Y. PARK, AND J. L. HIRSHFIELD

Abstract **—A unified single-mode theory is developed for the gyrotron traveling-wave amplifier (gyro-TWA) at harmonics of electron gyrofrequency, both in linear and nonlinear regimes. The theory is applicable to a wide class of waveguide cross sections and waveguide modes; it can also be useful for arbitrary harmonic numbers and with the generalized electron beam model. The waveguide fields are expanded into series of multipoles about the electron guiding centers. A general dispersion equation is derived. Some numerical examples of the gain–frequency curves of gyro-TWA's with out-ridged, magnetron-type, rectangular and circular waveguides are computed by employing the results of kinetic theory.**

I. Introduction

BECAUSE OF THEIR ABILITY to produce or to amplify millimeter and submillimeter waves at unprecedented power levels with high efficiency, gyrotron devices have been intensively investigated both theoretically and experimentally in the past two decades. Their promising applications include plasma heating, new millimeter- and submillimeter-wave radar systems, spectroscopy, and advanced accelerators.

This new class of microwave devices is based on the electromagnetic radiation mechanism known as the electron cyclotron maser instability, which originates from the electron azimuthal bunching due to the dependence of electron relativistic gyration frequency on energy.

A gyrotron device has an electron beam traveling within a waveguide (or one or more cavities) which is immersed in the applied magnetic field. Since the beam–field interaction takes place in the plane transverse to the direction of wave propagation, the electrons must have a substantial part of their kinetic energy in the form of gyration motion as they move on helical orbits along magnetic field lines. Fig. 1 illustrates the basic configuration of a gyrotron traveling-wave amplifier (gyro-TWA).

The electron cyclotron maser mechanism was recognized first by the astrophysicist R. Q. Twiss [1] in 1958. Shortly after Twiss's work, A. V. Gaponov [2] published a paper to describe the classical theory of cyclotron maser.

J. L. Hirshfield and J. M. Wachtel performed the first experiment that definitely demonstrated the existence of the electron cyclotron maser mechanism in 1964 [3], [4]. They reported an experiment with a 5-kV electron beam traveling along an axial magnetic field, and the beam was injected into a high-Q cylindrical cavity with most of the kinetic energy transverse to the applied magnetic field.

The early experiments were with small power and low efficiency, but since 1974 the advances in gyrotron research have come at a rapid pace. The advent of the intense pulsed relativistic electron beam renewed the interest in the cyclotron maser mechanism as a source of high-power microwave radiation. Powers of 800 MW at 4 cm [5], 350 MW at 2 cm [6], and 8 MW at 8 mm [7] have been generated with gyrotrons. Gyrotrons built by Soviet scientists have produced 1.25 MW of 45-GHz radiation with a pulse duration of 1 to 5 ms, 1.1 MW of 100-GHz radiation with a pulse duration of 100 μs, and 120 kW at 375 GHz with a pulse duration of 0.1 ms [8]. The efficiency of these gyrotron oscillators operating at the fundamental harmonic of gyration frequency is about 30–40 percent [9]. Jory and his group did an experiment which generated 200 kW CW at 60 GHz [10]. Temkin *et al.* reported 130 kW at 240 GHz with a pulse duration of 0.1 ms [11]. The experiments to heat plasmas in controlled-fusion devises have been done effectively [12]. Some new configurations for gyrotrons have been examined in the experiments. The gyro-TWA experiments have been performed and the results have surpassed those of the conventional TWA devices [13].

Hirshfield *et al.* [4] first employed plasma kinetic theory to analyze gyrotron interaction, and this approach has been widely used since. An electron distribution function (in space and momentum) is specified and the perturbed distribution function is obtained by integrating the linearized Vlasov equation along the unperturbed trajectories of the gyrating electrons. Another method of analyzing the interaction process is the Lagrangian formulation where one directly solves the equation of motion of the electrons in the applied and RF fields. If rigorous relativistic kinetics are required, such as in the high-power and the higher gyration harmonic gyrotron cases, the integration has to be accomplished numerically.

In developing gyrotrons at millimeter and submillimeter wavelengths for plasma heating, radar systems and some other purposes, there is an increasing necessity to reduce the weight and size of the devices and, consequently, to

Manuscript received December 18, 1985; revised June 14, 1986. This work was supported in part by the Office of Naval Research.

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IEEE Log Number 8610134.

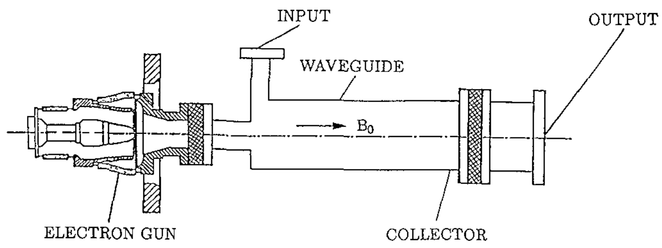

Fig. 1. Basic configuration of a gyro-TWA.

reduce the magnetic field substantially. For the amplification or generation of submillimeter waves with gyrotrons, an impractically high applied magnetic field would be required if the device is operated at the fundamental cyclotron harmonic. For a gyrotron operating at the same frequency range but at the sth harmonic of the electron gyrofrequency, the applied magnetic field is reduced approximately by a factor of s. This is of great importance, especially for uses in compact radar systems and for those applications where the device size and weight are critical to the system.

This paper presents a unified theory of gyro-TWA at harmonics of the electron gyrofrequency. The fields in the waveguide and, consequently, the forces on the electrons are expressed by infinite series of multipole components expanded around the axis of the electron helical trajectories. This makes the analyses, both linear and nonlinear, capable of handling several different shapes of waveguide cross sections for the RF structures of gyro-TWA devices.

The waveguide structure is very important for the operation of gyrotrons at higher harmonics of electron gyrofrequency. A good waveguide structure creates a suitable field pattern in the waveguide to enhance the beam–field interaction substantially at the operating gyration harmonic.

It has been realized in general that the RF-field transverse inhomogeneity in the waveguide is responsible for the interaction between the electron beam and the field in gyrotron devices at the harmonics of the electron gyration frequency [7]. In the present work, the analytical results show that the electron beam interaction with the field at the sth harmonic is associated with the multipole field of order $2s$ only if the field is expanded around the guiding centers of electrons. In order to achieve a good coupling between the waveguide field and electrons, a larger beam energy is required for gyrotrons working at higher gyration harmonics. However, unlike previous theoretical predictions, in this work it is shown that a higher order waveguide mode is not necessarily better for a gyrotron device to work at higher harmonics than a lower order mode, even though the higher waveguide mode has a higher transverse inhomogeneity in the RF field.

Starting with the Maxwell equations and the equation of motion of an electron, employing the weakly irregular waveguide theory, and expanding the waveguide field into an infinite series of the multipoles around the gyration centers of the electrons, we derive a set of equations in Section II to describe the nonlinear evolution of electron motion in a self-consistent manner. A dispersion equation is derived by iterating the solution of that set of equations. Section III is devoted to kinetic theory. The introduction of the Laplace transformation allows us to include the initial values. The small-signal gain–frequency relation is obtained through the inverse Laplace transformation. As examples, the small-signal gain–frequency curves for gyro-TWA's with several waveguide structures, such as the out-ridged, rectangular, magnetron-type, and circular waveguides are computed.

II. Nonlinear Theory

In this section, we formulate a set of basic equations for gyrotron nonlinear theory. Starting with the Maxwell equations and the equation of motion of an electron in the electromagnetic field, we derive a set of nonlinear equations which can be applied to the gyrotrons with different waveguides. A general dispersion equation of gyro-TWA is derived from that set of nonlinear equations with an iteration method.

In the analysis, all the RF fields are assumed to be time harmonic. From the Maxwell equations, an inhomogeneous Helmholtz equation for the magnetic field can be derived as

$$\nabla^2 \boldsymbol{B} + \frac{\omega^2}{c^2}\boldsymbol{B} = -\frac{4\pi}{c}\nabla \times \boldsymbol{J} \tag{1}$$

where ω is the angular frequency, c is the speed of light, and $\boldsymbol{J}$ is the current density.

For TE waveguide modes, $E_z = 0$, where z is along the axial direction of the waveguide. With the assumption $|d/dz \ln B_z| \ll 1$, we may write the axial component of the magnetic field in the following form:

$$B_z = F(z) B_z^0(\boldsymbol{r}_t) e^{j\omega t} \tag{2}$$

where $\boldsymbol{r}_t$ represents the transverse coordinates and $F(z)$ is, in general, a complex function of z. In the gyrotron analyses, $|d/dz \ln B_z| \ll 1$ is a good approximation since the beam–wave interaction is strong only in a frequency range that is close to the cutoff frequency of the operation wave mode, where the waveguide wavelength is long compared to the scale length of the beam–wave interaction.

If operator ∇ is written as $\nabla = \nabla_t + \boldsymbol{e}_z \partial/\partial z$, then $\nabla^2 = \nabla_t^2 + \partial^2/\partial_z^2$, and from (2) the equation for B_z takes the form

$$(\nabla_t^2 + k_c^2) F(z) B_z^0 + \left(\frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2} - k_c^2\right) F(z) B_z^0 = -\frac{4\pi}{c}(\nabla \times \boldsymbol{J}) \cdot \boldsymbol{e}_z. \tag{3}$$

A parameter k_c has been introduced in the above and it will be determined later. Since B_z is assumed to be in the form of (2) and the space charge effects will be neglected, the eigenvalues and the associated eigenfunctions of the waveguide for TE modes can be obtained by solving the equation

$$(\nabla_t^2 + k_c^2) B_z^0 = 0 \tag{4}$$

subject to the perfect conducting boundary condition on the wall of the waveguide

$$\boldsymbol{n}\cdot\nabla_t B_z^0 = 0 \tag{5}$$

where $\boldsymbol{n}$ is a unit vector normal to the waveguide wall surface.

From (3), for function $F(z)$, we have the following equation:

$$\left(\frac{d^2}{dz^2}+\frac{\omega^2}{c^2}-k_c^2\right)F(z) = -\frac{4\pi}{cN}\int_A (\nabla\times\boldsymbol{J})\cdot\boldsymbol{e}_z\left(B_z^0\right)^* dA \tag{6}$$

where

$$N = \int_A B_z^0\left(B_z^0\right)^* dA \tag{7}$$

$(B_z^0)^*$ is the complex conjugate of B_z^0, and the integration is over the cross section of the waveguide.

The transverse components of the electric and magnetic field for TE waveguide modes can be derived from the Maxwell equations as

$$\boldsymbol{B}_t = \frac{1}{k_c^2}\frac{\partial F(z)}{\partial_z}\nabla_t B_z^0 e^{j\omega t} \tag{8}$$

$$\boldsymbol{E}_t = \frac{j\omega}{ck_c^2}F(z)\boldsymbol{e}_z\times\nabla_t B_z^0 e^{j\omega t}. \tag{9}$$

In this formalism, the problem of determining the fields in the waveguide with moving electrons reduces to (4), which is the same equation as that for the empty waveguide, and to (6), which involves the electron beam and the fields in the waveguide. But for (6), we can have several different ways to obtain its right-hand side. In this work, two different approaches to obtain the right-hand side of (6) will be employed.

It should be pointed out that if the right-hand side of (6) is set to zero but k_c is assumed to be a function of z, then under single-mode assumption, (6) can be the basic equation for slow-varying waveguides used in gyrotrons [14]–[19].

The assumptions made in this nonlinear analysis include a single-wave mode, the neglect of space-charge effects, and an initially monoenergetic electron beam. In the tenuous beam case, the single waveguide mode is a very good description and has been confirmed by experiments [7], [10]. The beam and waveguide model is depicted in Fig. 2(a). In this model, the beam can be either annular or concentric. Moreover, since the shape of the waveguide cross section is not restricted to being circular or rectangular, it can be applicable to several different shapes. In the analyses, no assumption about beam energy is made; therefore, the results are valid even in the fully relativistic electron beam case. In the following, the field components in the waveguide are expanded into the series of Bessel functions. The coefficients in the expansions are dependent upon the waveguide structure.

Since the momentum $\boldsymbol{P} = \gamma m\boldsymbol{v}$, for the electrons in the electromagnetic field, the relativistic equations of motion

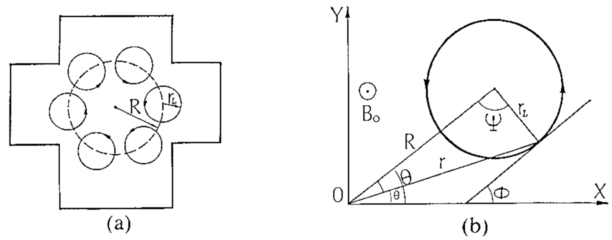

Fig 2. (a) Electron beam–waveguide model (b) Projection of an electron trajectory on the cross section of a waveguide.

are

$$\frac{d\boldsymbol{P}}{dt} = -e\left(\boldsymbol{E}+\frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right) \tag{10}$$

$$mc^2\frac{d\gamma}{dt} = -e\boldsymbol{v}\cdot\boldsymbol{E} \tag{11}$$

where $\gamma = (1-v^2/c^2)^{-1/2}$.

For TE modes, the fields in the waveguide can be expressed as $\boldsymbol{B} = (B_0+B_z)\boldsymbol{e}_z + B_x\boldsymbol{e}_x + B_y\boldsymbol{e}_y$, $\boldsymbol{E} = E_x\boldsymbol{e}_x + E_y\boldsymbol{e}_y$, and $B_0\boldsymbol{e}_z$ is the applied magnetic field. From the equation for γ in (11), it is seen that the energy exchange of the electrons for the TE waveguide modes is entirely from the interaction of electron transverse velocity with waveguide transverse electric field.

In general, we can expand the waveguide field into the infinite series in the following equation [32] in cylindrical coordinates:

$$B_z^0 = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} A_{nm}J_m(k_c r)e^{jm\theta}. \tag{12}$$

The coefficients A_{nm} in the series are dependent upon the shape of the waveguide cross section, $J_m(x)$ is the first-kind Bessel function of order m, and $r = (x^2+y^2)^{1/2}$. Setting $v = v_x + jv_y$, we have $v_x = 1/2(v + v^*)$, $v_y = -j/2(v - v^*)$, where v^* is the complex conjugate of v. Furthermore, if the solution of v is assumed to be in the form $v = v_t e^{j\Lambda}$, $v_t = (vv^*)^{1/2}$, and the phase angle $\Lambda = \tan^{-1}(v_y/v_x) = -\tan^{-1}[j(v-v^*)/v+v^*]$.

Following the approach of [20], we can have

$$\frac{d}{dt}\gamma = \frac{e\omega v_t}{2mc^3k_c}F(z)\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}A_{nm}J_{m-s}(k_cR)\cdot J_s'(k_cr_l)e^{j(\omega t - s\Psi + m\Theta)} \tag{13}$$

$$\frac{d}{dt}v_z = -\frac{ev_t}{2mc\gamma k_c}\left(\frac{\omega v_z}{c^2}+j\frac{\partial}{\partial_z}\right)F(z)\cdot\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}A_{nm}J_{m-s}(k_cR)\cdot J_s'(k_cr_l)e^{j(\omega t - s\Psi + m\Theta)} \tag{14}$$

$$\frac{dv}{dt} = -v\frac{d}{dt}\ln\gamma + \frac{e}{m\gamma}\left[\left(E_x+jE_y\right)+\frac{jv_z}{c}\left(B_x+jB_y\right)\right] - \frac{jeB_z^0}{mc\gamma}v - j\Omega_c v. \tag{15}$$

In Cartesian coordinates, by equating the real and the imaginary part on both sides of (15) correspondingly, we obtain the following two equations:

$$\dot{v}_t = -v_t\frac{d}{dt}\ln\gamma + \frac{e}{m\gamma}\left[\left(E_x - \frac{v_z}{c}B_y\right)\cos(\Lambda) + \left(E_y + \frac{v_z}{c}B_x\right)\sin(\Lambda)\right]$$
$$= -v_t\frac{d}{dt}\ln\gamma + \frac{e}{mc\gamma k_c^2}\left(\omega - v_z\frac{\partial}{\partial z}\right) \cdot F(z)\left[\sin(\Lambda)\frac{\partial}{\partial x} - \cos(\Lambda)\frac{\partial}{\partial y}\right]B_z^0 e^{j\omega t} \quad (16)$$

$$\dot{\Lambda} = -\frac{e}{m\gamma v_t}\left[\left(E_x - \frac{v_z}{c}B_y\right)\sin(\Lambda) + \left(E_y + \frac{v_z}{c}B_x\right)\cos(\Lambda)\right] + \frac{eB_z}{mc\gamma} + \Omega_c$$
$$= -\frac{e}{m\gamma c k_c^2 v_t}\left(\omega - v_z\frac{\partial}{\partial z}\right) \cdot F(z)\left[\cos(\Lambda)\frac{\partial}{\partial x} + \sin(\Lambda)\frac{\partial}{\partial y}\right]B_z^0 e^{j\omega t} + \frac{eB_z^0}{mc\gamma}F(z)e^{j\omega t} + \Omega_c \quad (17)$$

where $\Omega_c = eB_0/mc\gamma$.

With reference to Fig. 2(b), making use of Graf's addition theorem, we have

$$\dot{v}_t = -\frac{e}{2mc\gamma k_c}\left(\frac{\omega v_t^2}{c^2} - j\omega - v_z\frac{\partial}{\partial z}\right)F(z) \cdot \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} A_{nm}J_{m-s}(k_cR) \cdot J_s'(k_c r_l)e^{j(\omega t - s\Psi + m\Theta)} \quad (18)$$

$$\dot{\Lambda} = -\frac{e}{2mc\gamma k_c^2 v_t}\left(j\omega - v_z\frac{\partial}{\partial z} - k_c v_t\right)F(z) \cdot \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} A_{nm}J_{m-s}(k_cR) \cdot J_s(k_c r_l)e^{j(\omega t - s\Psi + m\Theta)} + \Omega_c \quad (19)$$

where $\dot{\Lambda}$ and $\dot{v}_t$ denote the time derivatives of Λ and v_t, respectively.

However, the function $F(z)$ involved in the above equations is still to be solved. We write the current density as

$$\boldsymbol{J} = -\frac{I_0}{v_{z0}}\boldsymbol{v} \quad (20)$$

where I_0 is the dc current. Making use of (8) and (9), the integral of the right-hand side of (6) can be cast into

$$\int_A dA\nabla \times \boldsymbol{J}\cdot\boldsymbol{e}_z(B_z^0)^* = \int_A dA\left[\nabla\cdot\left(\boldsymbol{J}\times(B_z^0)^*\boldsymbol{e}_z\right) + \nabla\times\left[(B_z^0)^*\boldsymbol{e}_z\right] + \nabla\times(B_z\boldsymbol{e}_z)\cdot\boldsymbol{J}\right]$$
$$= -\frac{jck_c^2 I_0}{\omega v_{z0}F(z)}\int_A dA\,\boldsymbol{v}\cdot\boldsymbol{E}^*. \quad (21)$$

It is seen from (21) that the axial component of the current does not contribute to the above integration and this integral is just proportional to the electron beam energy change rate averaged over the waveguide cross section. If the electron beam is idealized as having a single guiding center R, if the beam energy in most cases is not too high, and if k_z is small, we may approximate $v_z = v_{z0}$, and I_0/v_{z0} as the linear charge density in the waveguide. From (6) and (21), we can have the following equation:

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2} - k_c^2\right)F(z) - S = 0 \quad (22)$$

where

$$S = j\frac{2\pi I_0}{cN}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} A_{nm}\int_A dA\frac{v_t}{v_z}J_{m-s}(k_cR) \cdot J_s'(k_c r_l)e^{j(\omega t - s\Psi + m\Theta)}. \quad (23)$$

In the above equations, the Larmor radius $r_l = v_t/\Omega_c$. Therefore, we obtained a set of coupled nonlinear differential equations from (14), (18), (19), and (22). This set of equations describes the nonlinear evolution of electrons in the gyro-TWA devices. Essentially, this set of equations is a set of the particle orbit equations coupled, via the source term, to the inhomogeneous wave equation in a self-consistent manner. With the two-plate transmission-line and sheet-beam model, Zhurakhovskiy and Rapoport [20], [21], and later Sprangle [22], derived a set of equations to analyze the nonlinear evolution in the gyrotron devices. Fliflet *et al.* [23] used the method in [20] and [21] to carry out the formulation of the numerical nonlinear analysis with a more realistic circular waveguide and annular beam model. For nonlinear analyses, usually we integrate that set of the derived equations numerically. While nonlinear theory offers more information on the beam–field interaction behavior, which is especially necessary for the high-power gyrotron devices, the linear theory offers the basic physics of gyrotrons. Equation (22) is a secondary differential equation for one electron. If we solve this set of differential equations numerically and consider M electrons projected on the gyration circle in a unit length of the waveguide, we have to solve a system of order $5M$. But, if we assume that there is no reflection at the output end of the waveguide, the order of this system will be reduced to $5M-3$. For an unbunched "cold" electron beam, the initial values are the transverse velocity, the axial velocity, the initial phase angle, and the initial values of $F(0)$, $F'(0)$. If the initial phases are assumed to be uniformly distributed, we can specify $\Lambda_{0i} = 2i\pi/M$ for the ith electron ($i=1,2,\cdots,M$). By computing γ as a function of time, consequently as a function of z, and taking the average over phases and ensemble, the energy transfer efficiency from the electrons to the waveguide fields can be obtained, and the saturation process will be determined.

Leaving this ambitious task here, in what follows we derive the linear dispersion equation for gyrotrons by the iteration method.

If all the waveguide RF-field components are neglected, then we obtain the lowest order solution which corresponds to the motion of the electron in a uniform static magnetic field B_0, $v_t=v_{t0}$, $v_z=v_{z0}$, $\Lambda=(\omega/s-\Omega_c)\tau$, $\gamma=\gamma_0$; from (22) we have $F(z)=e^{-jk_z z}$. Here, the forward propagating wave is considered only because we assume that there is no reflecting wave at the output end of the waveguide, and $k_z=[(\omega/c)^2-k_c^2]^{1/2}$.

Substituting the zeroth order solution into the nonlinear equations and integrating over t, for a single harmonic number s, we obtain the solutions of the first-order approximation as

$$v_t=v_{t0}-\frac{e(\omega-k_z v_{z0})}{2mc\gamma k_c\Omega_s}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}A_{nm}\frac{v_{t0}}{V_{z0}}J_{m-s}(k_cR)\cdot J_s'(k_cr_l)\sin(\omega t-s\Psi-k_zz+m\Theta) \quad (24)$$

$$\gamma=\gamma_0-\frac{e\omega v_{t0}}{2mc^3k_c\Omega_s}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}A_{nm}J_{m-s}(k_cR)\cdot J_s'(k_cr_l)\cos(\omega t-s\Psi-k_zz+m\Theta) \quad (25)$$

$$v_z=v_{z0}+\frac{ev_{t0}k_z}{2mc\gamma_0k_c\Omega_s}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}A_{nm}J_{m-s}(k_cR)\cdot J_s'(k_cr_l)\sin(\omega t-s\Psi-k_zz+m\Theta) \quad (26)$$

$$z=v_{z0}\tau-\frac{ev_{t0}k_z}{2mc\gamma_0k_c\Omega_s^2}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}A_{nm}J_{m-s}(k_cR)\cdot J_s'(k_cr_l)\cos(\omega t-s\Psi-k_zz+m\Theta) \quad (27)$$

$$\Lambda\simeq\Omega_ct-\frac{e\omega\Omega_{c0}v_{t0}}{2m\gamma_0c^3k_c\Omega_s^2}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}A_{nm}J_{m-s}(k_cR)\cdot J_s'(k_cr_l)\cos(\omega t-s\Psi-k_zz+m\Theta)-\frac{e(\omega-k_zv_{z0}-k_cv_{t0})}{2mc\gamma_0k_c^2v_{t0}\Omega_s}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}A_{nm}J_{m-s}(k_cR)\cdot J_s(k_cr_l)\cos(\omega t-s\Psi-k_zz+m\Theta). \quad (28)$$

Making use of $s\Omega_{c0}\simeq\omega$ at cyclotron resonance and the above first-order approximations, considering that the second-order quantities are much smaller than the first-order quantities, and using the approximation $\dot\Psi\simeq\Omega_c$, $\sin\eta\simeq\eta$, $\cos\eta\simeq1$ for small angle η, we may write

$$\begin{aligned}\sin(\omega t-s\Psi-k_zz+m\Theta)&\simeq\sin(\Omega_st+m\Theta)-\left(\frac{\omega^2}{c^2}-k_z^2\right)\frac{ev_{t0}}{2mc\gamma_0k_c\Omega_s^2}\\&\cdot\cos^2(\Omega_st+m\Theta)\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}A_{nm}J_{m-s}(k_cR)J_s'(k_cr_l)\\&-\frac{e(\omega-k_zv_z-k_cv_{t0})}{2mc\gamma_0k_cv_{t0}\Omega_s}\\&\cdot\cos^2(\Omega_st+m\Theta)\\&\cdot\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}A_{nm}J_{m-s}(k_cR)J_s(k_cr_l).\end{aligned} \quad (29)$$

In the above equations, $\Omega_s=\omega-s\Omega_{c0}-k_{z0}v_{z0}$.

Substituting the above first-order approximation of v_t, v_z and (29) into (13) and keeping the second-order approximation terms only, we obtain

$$\begin{aligned}\frac{1}{v_z}\frac{d\gamma}{dt}=&\frac{e\omega v_{t0}}{2mc^3v_{z0}k_c}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}A_{nm}J_{m-s}(k_cR)J_s'(k_cr_l)\sin(\Omega_st+m\Theta)-\frac{e^2\omega}{4m^2v_{z0}c^4k_c^2\gamma_0}\left\{\left(\frac{\omega^2}{c^2}-k_z^2\right)\frac{v_{t0}^2}{\Omega_s^2}\right.\\&\cdot\cos^2(\Omega_st+m\Theta)\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}[A_{nm}]^2[J_{m-s}(k_cR)J_s'(k_cr_l)]^2-\frac{(\omega-k_zv_{z0})}{\Omega_s}\\&\cdot\sin^2(\Omega_st+m\Theta)\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}[A_{nm}]^2[J_{m-s}(k_cR)J_s'(k_cr_l)]^2+\frac{(\omega-k_zv_{z0}-k_cv_{t0})}{\Omega_s}\cos^2(\Omega_st+m\Theta)\\&\left.\cdot\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}[A_{nm}]^2J_{m-s}^2(k_cR)J_s'(k_cr_l)J_s(k_cr_l)\right\}.\end{aligned} \quad (30)$$

The first term in the large braces in (30) is proportional to v_{t0}^2. There are two parts in this term. If $k_z=0$, then this term is entirely due to the transverse force by the transverse field components and derives from the electron cyclotron maser instability. The other part, which is proportional to k_z^2, is due to the force in the z direction, that is, due to the Weible instability. This Weible part causes a change of the phase velocity of the electron motion. This can be made clear by observing (26). If $k_z=0$ or $v_{t0}=0$, then $v_z=v_{z0}$ and the Weibel instability would disappear. Thus, a conclusion we can make is that the existence of the Weibel instability is always associated with the electron transverse motion and with the propagating RF wave in the waveguide. The electron cyclotron maser part and the Weibel part always oppose each other, since their signs are different from each other. When $\omega^2>c^2k_z^2$, the cyclotron resonance instability dominates; otherwise, the Weibel instability dominates. When $k_z\neq0$, there is a frequency shift that causes the amplification or oscillation frequency to shift away from $s\Omega_{c0}$. We have seen that the energy change of the electrons is completely due to the interaction between the transverse field and the velocity for TE mode interaction. The second term in the big braces is due to the change of the transverse velocity and is proportional to the inverse of Ω_s; while the third term, which is also inversely proportional to Ω_s, is associated with the change of the gyration phase velocity. These two terms set a threshold for the instability. The right-hand side of (6) may be written in the form

$$\int_A dA \nabla \times \boldsymbol{J} \cdot \boldsymbol{e}_z (B_z^0)^* = \frac{ck_c^2}{\omega} \int_A dA \left(\frac{I_0}{v_z} \right) \boldsymbol{v} \cdot \boldsymbol{E}^* = -\frac{jmc^3 k_c^2 I_0}{e\omega F(z)} \int_A dA \frac{1}{v_z} \frac{d\gamma}{dt}. \quad (31)$$

Substituting (30) into (31) and taking the average over a period of a slow time variable $\Omega_s t$ and $m\Theta$, from (22) we obtain the dispersion equation as

$$\frac{\omega^2}{c^2} - k_z^2 - k_c^2 = \frac{e\pi I_0}{Nmc^2 \gamma_0 v_{z0}} \left\{ \frac{\beta_t^2(\omega^2 - k_z^2 c^2)}{\Omega_s^2} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} [A_{nm}]^2 [J_{m-s}(k_c R) J_s'(k_c r_l)]^2 - \frac{\omega - k_z v_{z0}}{\Omega_s} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} [A_{nm}]^2 [J_{m-s}(k_c R) J_s'(k_c r_l)]^2 + \frac{\omega - k_z v_z - k_c v_{t0}}{\Omega_s} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} [A_{nm}]^2 J_{m-s}^2(k_c R) J_s'(k_c r_l) J_s(k_c r_l) \right\} \quad (32)$$

where $\beta_t = v_t/c$.

From (32), the dispersion equation can be derived in the form

$$\frac{\omega^2}{c^2} - k_z^2 - k_c^2 = \frac{e\pi I_0}{Nmc^2 \gamma_0 v_{z0}} \left[\frac{\beta_t^2(\omega^2 - k_z^2 c^2)}{\Omega_s^2} H - \frac{(\omega - k_z v_{z0})}{\Omega_s} Q \right] \quad (33)$$

where

$$H = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} [A_{nm}]^2 [J_{m-s}(k_c R) J_s'(k_c r_l)]^2 \quad (34)$$

$$Q = H - \left(1 - \frac{k_c v_{t0}}{\omega - k_z v_{z0}} \right) \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} [A_{nm}]^2 J_{m-s}^2(k_c R) \cdot J_s'(k_c r_l) J_s(k_c r_l). \quad (35)$$

In the circular waveguide case, for the TE_{nm} mode, $k_c = p_{nm}/a$, where a is the radius of the waveguide and p_{nm} is the mth root of the Bessel function $J_n(x)$. In the above dispersion equation, the coefficient A_{nm} equals unity and there is no summation involved. Comparing this dispersion equation with that derived by Chu *et al.* [24], for the circular waveguide TE_{nm} mode, we find that the function H is the same as H_{sm} in [24]. From the Bessel equation, the first term in the function Q in (34) can be made the same as that in [24]. However, there is some difference between the rest of Q in (35) and the rest in the function Q_{sm} in [24]. The difference is due to the different approximations made in the two different approaches to derive the dispersion equation. As pointed out in [24], in the dispersion equation, the term with Q_{sm} involved imposes a threshold beam energy for the instability. At higher harmonics, that term is very small compared to the term proportional to H and it can be neglected.

III. Gyrotron Kinetic Theory

With the field components written in the forms of (2), (8), and (9), in Section II from nonlinear theory we derived an equation for $F(z)$ in (22) which has to be solved simultaneously with other equations. In this section, we employ plasma kinetic theory to solve this equation to get $F(z)$; moreover, we shall obtain the small-signal gain–frequency relation for gyro-TWA's.

Because of its relative simplicity and the case with which the physical results obtained can be understood, kinetic theory has been widely used in gyrotron analysis. As a standard approach in plasma physics, the linearized Vlasov equation is solved by the method of characteristics, and the initial value may be included in introducing a Laplace transformation. Through the inverse Laplace transformation, the function $F(z)$ which describes the profile of the RF field along the waveguide with the presence of moving electrons is obtained, and this allows the gain of the power flow of the device to be computed. Park *et al.* [25] have used this approach to analyze the slow-wave gyrotron amplifiers for the circular electrical waveguide modes. However, the analysis presented in this section is with the generalized waveguide–beam model shown in Fig. 2(a) and is for any TE waveguide modes.

As usual, we find just the first-order perturbation of the electron distribution function. Therefore, this is a linear theory. In doing this analysis, several assumptions are made. First, it is assumed that the space-charge effect can be neglected; second, the electron beam and the RF wave in the waveguide are described by the linearized Maxwell-Vlasov equations; third, this is a single-mode analysis, and coupling with the neighboring wave modes is assumed to be negligible. The nature of the electron helical motion in the waveguide makes the cylindrical coordinate system most suitable for this analysis. But, since this analysis is a generalized one and is applicable to gyro-TWA devices with different shapes of the waveguide cross section, this analysis is carried out in the Cartesian coordinate system first; then a transformation to cylindrical coordinates is made naturally by using some Bessel function identities.

In addition to the Maxwell equations, the Vlasov equation

$$\frac{\partial f}{\partial t}+\frac{\boldsymbol{u}}{\gamma}\cdot\nabla f-\frac{e}{m}\left(E+\frac{1}{c\gamma}\boldsymbol{u}\times\boldsymbol{B}\right)\cdot\nabla_u f=0 \tag{36}$$

and two coupling equations

$$\boldsymbol{J}=-e\int d^3u\frac{\boldsymbol{u}}{\gamma}f \tag{37}$$

$$\rho=-e\int d^3uf \tag{38}$$

together form the basic equations of the kinetic theory. Here, $f(\boldsymbol{u}, \boldsymbol{r}, t)$ is the electron distribution function in momentum, space, and time; $\boldsymbol{u}=\gamma\dot{\boldsymbol{r}}$, $\gamma=(1+u^2/c^2)^{1/2}$, and $\dot{\boldsymbol{r}}$ stands for the time derivative of $\boldsymbol{r}$. With assumptions $|f_1|\ll|f_0|$ and $|B_1|\ll|B_0|$, by setting $f=f_0+f_1$, $\boldsymbol{E}=\boldsymbol{E}_1$, and $\boldsymbol{B}=\boldsymbol{B}_0+\boldsymbol{B}_1$, the perturbed electron distribution function can be obtained by the method of characteristics, viz, by integrating the equation along the unperturbed electron trajectory.

With reference to Fig. 2(b), we write $\boldsymbol{u}=u_t\boldsymbol{e}_t+u_z\boldsymbol{e}_z$, $\boldsymbol{e}_t=\boldsymbol{e}_x\cos\Phi+\boldsymbol{e}_y\sin\Phi$, and $u_t=(u_x^2+u_y^2)^{1/2}$. For R, the guiding center of electrons, we have

$$\nabla_u R=-\frac{1}{\Omega_c}\boldsymbol{e}_\Theta$$
$$=\frac{1}{\Omega_c}(\boldsymbol{e}_x\sin\Theta-\boldsymbol{e}_y\cos\Theta). \tag{39}$$

Furthermore, we may write

$$\nabla_u f_0=\boldsymbol{e}_z\frac{\partial f_0}{\partial u_z}+\boldsymbol{e}_t\frac{\partial f_0}{\partial u_t}+\frac{\partial f_0}{\partial R}\nabla_u R$$
$$=\boldsymbol{e}_z\frac{\partial f_0}{\partial u_z}+\boldsymbol{e}_x\left(\cos\Phi\frac{\partial f_0}{\partial u_t}+\frac{1}{\Omega_c}\sin\Theta\frac{\partial f_0}{\partial R}\right)$$
$$+\boldsymbol{e}_y\left(\sin\Phi\frac{\partial f_0}{\partial u_t}-\frac{1}{\Omega_c}\cos\Theta\frac{\partial f_0}{\partial R}\right) \tag{40}$$

where $\Omega_c=eB_0/mc\gamma$ is the electron relativistic gyration frequency.

The waveguide field components in the beam–field interaction region can be written in the form of (8), (9), and (12), both the coefficients in the series are dependent upon the geometry of the waveguide cross section and are derived in [32]. We may write

$$\left[\boldsymbol{E}_1+\frac{\boldsymbol{u}}{c\gamma}\times\boldsymbol{B}_1\right]\cdot\nabla_u f_0=\frac{1}{ck_c^2}\left\{\left[\frac{1}{\gamma}\frac{\partial F(z)}{\partial z}\left(u_t\frac{\partial f_0}{\partial u_z}-u_z\frac{\partial f_0}{\partial u_t}\right)+j\omega F(z)\frac{\partial f_0}{\partial u_t}\right]\left(\sin\Phi\frac{\partial}{\partial x}-\cos\Phi\frac{\partial}{\partial y}\right)B_{1z}^0\right.$$
$$\left.+\left(j\omega-\frac{u_z}{\gamma}\frac{\partial}{\partial z}\right)F(z)\frac{1}{\Omega_c}\frac{\partial f_0}{\partial R}\left(\cos\Theta\frac{\partial}{\partial x}+\sin\Theta\frac{\partial}{\partial y}\right)B_{1z}^0+\frac{u_tk_c^2}{\gamma}F(z)\sin\Psi\frac{1}{\Omega_c}\frac{\partial f_0}{\partial R}B_{1z}^0\right\}. \tag{41}$$

Setting $k_{x1n}=k_c\sin\lambda_n$, $k_{y1n}=k_c\cos\lambda_n$, using the Bessel identities

$$e^{jk_cr\sin(\theta+\lambda_n)}=\sum_{m=-\infty}^{\infty}J_m(k_cr)e^{jm(\theta+\lambda_n)} \tag{42a}$$

$$e^{-jk_cr\sin(\theta+\lambda_n)}=\sum_{m=-\infty}^{\infty}(-1)^mJ_m(k_cr)e^{jm(\theta+\lambda_n)} \tag{42b}$$

and noting the relations from Fig. 2(b) that for electron orbit at equilibrium $\Psi'=\Psi-\Omega_c(z-z')/v_z$ and $t-t'=(z-z')/v_z$, we obtain the integral of the perturbed distribution function

$$f_1=\frac{e}{m}\int_{t-z/v_z}^{t}dt'e^{-j\omega t'}\left(\boldsymbol{E}_1+\frac{\boldsymbol{u}}{c\gamma}\times\boldsymbol{B}_1\right)\cdot\nabla_u f_0$$
$$=\frac{eN_e}{mck_c}e^{j\omega t}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}A_{nm}e^{jm\Theta}$$
$$\cdot\int_0^z dz'G(z-z')Q(z') \tag{43}$$

where N_e is the electron number per unit axial length, and

$$G(z-z')=\frac{1}{v_z}e^{j(\omega-s\Omega_c)z-z'/v_z} \tag{44}$$

$$Q(z)=\left[\left(j\omega F-\frac{u_z}{\gamma}\frac{\partial F}{\partial z}\right)\frac{\partial f_0}{\partial u_t}+\frac{u_t}{\gamma}\frac{\partial F}{\partial z}\frac{\partial f_0}{\partial u_z}\right]J_{m-s}(k_cR)$$
$$\cdot J_s'(k_cr_l)+\left(j\omega F-\frac{u_z}{\gamma}\frac{\partial F}{\partial z}-js\frac{\Omega_c}{\gamma}F\right)J_{m-s}'(k_cR)$$
$$\cdot J_s(k_cr_l)\frac{1}{\Omega_c}\frac{\partial f_0}{\partial R}+\frac{jk_c^2u_t}{2\gamma}F\left[J_{m-s-1}(k_cR)\right.$$
$$\left.+J_{m-s+1}(k_cR)\right]J_s'(k_cr_l)\frac{1}{\Omega_c}\frac{\partial f_0}{\partial R}. \tag{45}$$

The Laplace transformation $\bar{F}(K)$ of function $F(z)$ is defined as

$$\bar{F}(K)=\int_0^\infty F(z)e^{-jKz}dz \tag{46}$$

where the imaginary part of K is chosen to be positive and large enough in magnitude so that the convergence of the integral is guaranteed.

Following the standard procedure, we obtain the transformation of $G(z)$ in (44)

$$\bar{G}(K)=\int_0^\infty dz\,e^{-j(\omega-Ku_z/\gamma-s\Omega_c)z}$$
$$=j\frac{1}{\Omega_s(K)} \tag{47}$$

where $\Omega_s(K)=\omega-Ku_z/\gamma-s\Omega_c$. The Laplace transformation of $Q(z)$ in (45) is given by

$$\bar{Q}(K) = \bar{Q}_1(K) + \bar{Q}_0(K) \tag{48}$$

where

$$\bar{Q}_1(K) = j\bar{F}(K)\left\{\left[\left(\omega - \frac{Ku_z}{\gamma}\right)\frac{\partial f_0}{\partial u_t} + \frac{Ku_t}{\gamma}\frac{\partial f_0}{\partial u_z}\right] J_{m-s}(k_cR)J_s'(K_c r_l) + \Omega_s(K) J'_{m-s}(k_cR)J_s(k_c r_l)\frac{1}{\Omega_c}\frac{\partial f_0}{\partial R} \right.$$
$$\left. + \frac{1}{2}\frac{k_c^2 u_t}{\gamma}\left[J'_{m-s-1}(k_cR) + J_{m-s+1}(k_cR)\right] J_s'(k_c r_l)\frac{1}{\Omega_c}\frac{\partial f_0}{\partial R}\right\} \tag{49}$$

$$\bar{Q}_0(K) = F(0)\left\{\left(\frac{u_z}{\gamma}\frac{\partial f_0}{\partial u_t} - \frac{u_t}{\gamma}\frac{\partial f_0}{\partial u_z}\right) J_{m-s}(k_cR)J_s'(k_c r_l) + \frac{u_z}{\gamma}J'_{m-s}(k_cR)J_s(k_c r_l)\frac{1}{\Omega_c}\frac{\partial f_0}{\partial R}\right\}. \tag{50}$$

Making use of the convolution theorem, the Laplace transformation of the perturbed electron distribution function is obtained as

$$\bar{f}_1(K) = e^{j\omega t}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\frac{eN_eA_{nm}}{mck_c}e^{js\Psi}\bar{G}(K)\bar{Q}(K) \tag{51}$$

where $\bar{G}(K)$ and $\bar{Q}(K)$ are given by (47) and (48), respectively. In the following, we derive the perturbed current density in the waveguide. With this formalism of analyses, the axial component of the perturbed current does not contribute to the gyrotron interaction for TE wave modes, so it will be ignored.

The Laplace transformation of the perturbed current density is defined as

$$\bar{J}(K) = \bar{J}_\theta(K)\boldsymbol{e}_\theta + \bar{J}_r(K)\boldsymbol{e}_r. \tag{52}$$

Introducing the two quantities

$$\bar{J}_c(K) = \bar{J}_\theta(K) + j\bar{J}_r(K) \tag{53a}$$

$$\bar{J}_c^\dagger(K) = \bar{J}_\theta(K) - j\bar{J}_r(K) \tag{53b}$$

we have

$$\bar{J}_c(K) = -e\int d^3u\frac{u_t}{\gamma}e^{j\xi}\bar{f}_1(K) \tag{54a}$$

$$\bar{J}_c^\dagger(K) = -e\int d^3u\frac{u_t}{\gamma}e^{-j\xi}\bar{f}_1(K). \tag{54b}$$

We obtain the Laplace transformation of (6) as

$$\left(\frac{\omega^2}{c^2} - k_c^2 - K^2\right)\bar{F}(K) - jKF(0) - \frac{d}{dz}F(0)$$
$$= \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\frac{4\pi A_{nm}}{Nc}\int_0^{2\pi}d\theta\int_0^{R_\omega}r\,dr\left\{\frac{\partial}{\partial r}\left[rJ_r(K)\right] - \frac{\partial}{\partial\theta}J_\theta(K)\right\}J_m(k_cr)$$
$$= -\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\frac{4\pi A_{nm}}{Nc}\int_0^{2\pi}d\theta\int_0^{R_\omega}dr\,k_cr\left[\bar{J}_c(K)J_{m-1}(k_cr) - \bar{J}_c^\dagger(K)J_{m+1}(k_cr)\right]. \tag{55}$$

In the last step in writing (55), integration by parts has been used. Then, from (53), we have

$$\bar{J}_c(K) = -e\int d^3u\,e^{j\xi}\frac{u_t}{\gamma}\bar{f}_1(K)$$
$$= -e^{j\omega t}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\frac{N_ee^2A_{nm}}{mck_c}e^{-jn\Theta}\cdot\int du_t\,du_z\,d\xi\,e^{js\Psi}e^{j\xi}\frac{u_t^2}{\gamma}\bar{G}(K)\bar{Q}(K) \tag{56a}$$

$$\bar{J}_c^\dagger(K) = -e\int d^3u\,e^{-j\xi}\frac{u_t}{\gamma}\bar{f}_1(K)$$
$$= -e^{j\omega t}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\frac{N_ee^2A_{nm}}{mck_c}e^{-jn\Theta}\cdot\int du_t\,du_z\,d\xi\,e^{js\Psi}e^{-j\xi}\frac{u_t^2}{\gamma}\bar{G}(K)\bar{Q}(K) \tag{56b}$$

where $\bar{Q}(K)$ is given by (48).

To evaluate $\bar{J}_c(K)$ and $\bar{J}_c^\dagger(K)$, the electron distribution function has to be specified. In order for the distribution function to be general, we write it in the following form:

$$f_0(u_z, u_t, R) = f_c(u_z^0, u_t^0, R^0)f_s(u_z, u_t, R) \tag{57}$$

$$f_c(u_z^0, u_t^0, R^0) = f_u(u_z^0, u_t^0)f_R(R^0). \tag{58}$$

Furthermore, we write

$$f_u = \frac{1}{2\pi u_t}\delta(u_t - u_t^0)\delta(u_z - u_z^0) \tag{59}$$

$$f_R = \frac{1}{2\pi R}\delta(R - R_0) = \frac{1}{2\pi r r_l\sin\xi}\left[\delta(\xi - \xi_0) - \delta(\xi + \xi_0)\right]\cdot\Delta(r) \tag{60}$$

where

$$\Delta(r)=\begin{cases}1, & r_-\le r\le r_+\\ 0, & \text{otherwise}\end{cases}$$

$$r_\pm = R_0 \pm r_l$$

$r_l = u_t/\Omega_c$ is the electron Larmor radius in the applied magnetic field, and f_s is an arbitrary function of u_t, u_z, and R. The Dirac delta distribution function f_c represents the "cold" beam with infinitely thin guiding center distribution and is normalized to be an electron per unit length. For "cold" electron beam distribution function, $f_0 = f_c$. After performing the integrals, we obtain $\bar{J}_c(K)$ as

$$\bar{J}_c(K) = e^{j\omega t}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\frac{N_e e^2 A_{nm}}{2\pi mc^2 k_c u_t}\cdot[Q_1(K) - jQ_0(K)] \quad (61)$$

where

$$Q_1(K) = \bar{F}(K)\left\{\left[\frac{Ku_t}{\gamma}\frac{\partial f_u}{\partial u_z} + \left(\omega - \frac{Ku_z}{\gamma}\right)\frac{\partial f_u}{\partial u_t}\right]\cdot\left(\Phi + \frac{u_t^2}{\gamma\Omega_s(K)}\right) + \frac{u_t^2}{2\gamma\Omega_c}J_s(k_c r_l)\Psi_- + \frac{u_t^2 k_c r_l}{2\gamma\Omega_s(K)}J_s'(k_c r_l)\Psi_+\right\} \quad (62)$$

$$Q_0(K) = F(0)\left[\left(\frac{u_t}{\gamma}\frac{\partial}{\partial u_z} - \frac{u_z}{\gamma}\frac{\partial}{\partial u_t}\right)\frac{u_t^2}{\gamma\Omega_s(K)}J_s'(k_c r_l)\Phi_+ - \frac{u_t^2 u_z}{2\gamma\Omega_c}J_s(k_c r_l)\Psi_-\right]. \quad (63)$$

$\bar{J}_c^\dagger(K)$ is given by replacing Ψ_-, Ψ_+, Φ_+ with Ψ_{c-}, Ψ_{c+}, Φ_- in the expression of $\bar{J}_c(K)$ correspondingly. In the above

$$\Phi_\pm = e^{-jm\Psi}\frac{1}{\pi r r_l^2 \sin\xi}J_{m-s}(k_c R)\cos[(m-s)\Psi \pm \xi] \quad (64)$$

$$\begin{aligned}\Psi_- = e^{-jm\Psi}\frac{1}{\pi r r_l^2 \sin\xi}\cdot\Big\{&-k_c r_l J_{m-s}(k_c R)\cos[(m-s)\Psi+\xi]\\ &-\frac{1}{2}k_c r_l J_{m-s+2}(k_c R)\cos[(m-s+2)\Psi+\xi]\\ &-\frac{1}{2}k_c r_l J_{m-s-2}(k_c R)\cos[(m-s-2)\Psi+\xi]\\ &+(m-s+1)J_{m-s+1}(k_c R)\cos[(m-s+1)\Psi+\xi]\\ &+(m-s+1)J_{m-s-1}(k_c R)\cos[(m-s-1)\Psi+\xi]\Big\}\end{aligned} \quad (65)$$

$$\begin{aligned}\Psi_+ = e^{-jm\Psi}\frac{1}{\pi r r_l^2 \sin\xi}\cdot\Big\{&\frac{1}{2}k_c r_l J_{m-s+2}(k_c R)\cos[(m-s+2)\Psi+\xi]\\ &-\frac{1}{2}k_c r_l J_{m-s-2}(k_c R)\cos[(m-s-2)\Psi+\xi]\\ &-(m-s+1)J_{m-s+1}(k_c R)\cos[(m-s+1)\Psi+\xi]\\ &+(m-s+1)J_{m-s-1}(k_c R)\cos[(m-s-1)\Psi+\xi]\Big\}\end{aligned} \quad (66)$$

$$\begin{aligned}\Psi_{c-} = e^{-jm\Psi}\frac{1}{\pi r r_l^2 \sin\xi}\cdot\Big\{&-k_c r_l J_{m-s}(k_c R)\cos[(m-s)\Psi-\xi]\\ &-\frac{1}{2}k_c r_l J_{m-s+2}(k_c R)\cos[(m-s+2)\Psi-\xi]\\ &-\frac{1}{2}k_c r_l J_{m-s-2}(k_c R)\cos[(m-s-2)\Psi+\xi]\\ &+(m-s-1)J_{m-s-1}(k_c R)\cos[(m-s-1)\Psi-\xi]\\ &+(m-s-1)J_{m-s+1}(k_c R)\cos[(m-s+1)\Psi-\xi]\Big\}\end{aligned} \quad (67)$$

$$\begin{aligned}\Psi_{c+} = e^{-jm\Psi}\frac{1}{\pi r r_l^2 \sin\xi}\cdot\Big\{&\frac{1}{2}k_c r_l J_{m-s+2}(k_c R)\cos[(m-s+2)\Psi-\xi]\\ &-\frac{1}{2}k_c r_l J_{m-s-2}(k_c R)\cos[(m-s-2)\Psi-\xi]\\ &+(m-s-1)J_{m-s-1}(k_c R)\cos[(m-s-1)\Psi-\xi]\\ &-(m-s-1)J_{m-s+1}(k_c R)\cos[(m-s+1)\Psi-\xi]\Big\}.\end{aligned} \quad (68)$$

Now we shall work out the integral over the waveguide cross section on the right-hand side of (55). In cylindrical coordinates, the integral is over r and θ. Making use of the relation $dr = r_l \sin\xi\, d\Psi$, we may convert the integral over r into an integral over Ψ. After evaluating the integral over Ψ and θ, we get

$$\left(\frac{\omega^2}{c^2} - k_c^2 - K^2\right)\bar{F}(K) - jKF(0) - \frac{d}{dz}F(0) = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\frac{e^2 N_e}{mc^2}\frac{\pi A_{nm}^2}{N k_c u_t}\cdot[\bar{F}(K)T_1(K) - jF(0)T_0(K)] \quad (69)$$

where

$$T_1(K)=\left[\frac{Ku_t}{\gamma}\frac{\partial}{\partial u_z}+\left(\omega-\frac{Ku_z}{\gamma}\right)\frac{\partial}{\partial u_t}\right]\cdot\left[J_s'(k_cr_l)\frac{u_t^2}{\gamma\Omega_s(K)}\hat{\Phi}_+\right]+\frac{u_t^2}{2\gamma\Omega_c}J_s(k_cr_l)\hat{\Psi}_-+\frac{u_t^3k_c}{2\gamma^2\Omega_c\Omega_s(K)}J_s(k_cr_l)\hat{\Psi}_+ \tag{70}$$

$$T_0(K)=\left[\frac{u_t}{\gamma}\frac{\partial}{\partial u_z}-\frac{u_z}{\gamma}\frac{\partial}{\partial u_t}\right]J_s'(k_cr_l)\frac{u_t^2}{\gamma\Omega_s(K)}\hat{\Phi}_+-\frac{u_tu_z}{2\gamma^2\Omega_c}J_s(k_cr_l)\hat{\Psi}_- \tag{71}$$

and

$$\hat{\Phi}_+=k_cJ_{m-s}^2(k_cR)J_s'(k_cr_l) \tag{72}$$

$$\begin{aligned}\hat{\Psi}_-=\frac{2k_c^2}{k_cr_l}\Bigg\{&-J_{m-s}^2(k_cR)J_s'(k_cr_l)+\left[(m-s)J_{s+1}'(k_cr_l)+\frac{s+1}{k_cr_l}J_{s+1}(k_cr_l)\right]J_{m-(s+1)}(k_cR)\\&+\left[(m-s)J_{s-1}'(k_cr_l)+\frac{s-1}{k_cr_l}J_{s-1}(k_cr_l)\right]J_{m-(s-1)}(k_cR)\\&-\frac{1}{2}k_cr_lJ_{s+2}'(k_cr_l)J_{m-(s+2)}^2(k_cR)-\frac{1}{2}K_cr_lJ_{s-2}'(k_cr_l)J_{m-(s-2)}^2(k_cR)\Bigg\}\end{aligned} \tag{73}$$

$$\begin{aligned}\hat{\Psi}_+=-\frac{2k_c^2}{k_cr_l}\Bigg\{&\left[(m-s)J_{s+1}'(k_cr_l)+\frac{s+1}{k_cr_l}J_{s+1}(k_cr_l)\right]J_{m-(s+1)}(k_cR)\\&-\left[(m-s)J_{s-1}'(k_cr_l)+\frac{(s-1)}{k_cr_l}J_{s-1}(k_cr_l)\right]J_{m-(s-1)}(k_cR)\\&-\frac{1}{2}k_cr_lJ_{s+2}'(k_cr_l)J_{m-(s+2)}^2(k_cR)+\frac{1}{2}k_cr_lJ_{s-2}'(k_cr_l)J_{m-(s-2)}^2(k_cR)\Bigg\}.\end{aligned} \tag{74}$$

Finally, we obtain $\bar{F}(K)$ in a quotient form

$$\bar{F}(K)=\frac{N(K)}{D(K)} \tag{75}$$

where

$$D(K)=\left(\frac{\omega^2}{c^2}-K^2-k_c^2\right)\Omega_s^2(K)-\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\frac{\pi\nu A_{nm}^2}{\gamma N}D_{10}(K) \tag{76}$$

$$N(K)=jF(0)\left[K-\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\frac{\pi\nu A_{nm}^2}{\gamma N}D_{00}(K)\right] \tag{77}$$

where $\nu=N_ee^2/mc^2$ is the Budker parameter and

$$\begin{aligned}D_{10}(K)=&\left(K^2-\frac{\omega^2}{c^2}\right)\frac{u_t^2}{c^2}\left[J_s'(k_cr_l)j_{m-s}(k_cR)\right]^2\\&-k_cr_l\Omega_s(K)\left[2\Omega_c\left(1-\frac{s^2}{k_c^2r_l^2}\right)J_s(k_cr_l)J_{m-s}^2(k_cR)-\frac{k_cu_t}{2}\hat{\Psi}_+\right]J_s'(k_cr_l)\\&-\Omega_s^2(K)\left[2\gamma k_cr_l\left(1-\frac{s^2}{k_c^2r_l^2}\right)J_s'(k_cr_l)J_{m-s}^2(k_cR)-\frac{k_cu_t}{2}\hat{\Psi}_-\right]J_s(k_cr_l)\end{aligned} \tag{78}$$

$$\begin{aligned}D_{00}(K)=\frac{u_t}{\gamma\Omega_s^2(K)}\Bigg\{&\frac{Ku_t^2}{\gamma^2}\left[J_s'(k_cr_l)J_{m-s}(k_cR)\right]^2\\&+\frac{2k_cr_lu_z}{\gamma}\Omega_s(K)J_s(k_cr_l)J_s'(k_cr_l)J_{m-s}^2(k_cR)-\frac{k_cu_t}{2\Omega_c}\Omega_s^2(K)J_s(k_cr_l)\hat{\Psi}_-\Bigg\}.\end{aligned} \tag{79}$$

Similar to the commonly used beam–field coupling coefficient, for a single gyration harmonic number s, we define a beam–field coupling coefficient

$$H_c = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (A_{nm})^2 [J_s'(k_c r_l) J_{m-s}(k_c R)]^2. \quad (80)$$

In the simple circular waveguide case, (80) reduces to the common form

$$H_c' = [J_s'(k_c r_l) J_{m-s}(k_c R)]^2 \quad (81)$$

used in many papers [24], [27], [28]. A good physical explanation of the beam–field coupling coefficient can be found in [28].

In the expression of D_{10} in (78) the first term, which is proportional to H_c, is the source term; the second and the third terms impose a threshold on the instability. From this analytical result, if H_c vanishes, clearly both the cyclotron maser instability and the Weibel instability will vanish.

Because $|J_0(x)| \leq 1$, $|J_m(x)| \leq 1/\sqrt{2}$, we many conclude that for the same beam energy, the same waveguide structure, the same wave mode, and the same frequency range, a concentric beam has a bigger coupling with the fields than an annular beam. This offers a simple explanation for the utilization of rotating electron layers instead of annular beams in most of the reported microwave generation experiments at higher harmonics of gyration frequency. In the case of a concentric electron beam, $R = 0$, $H_c \neq 0$ only if $m = s$. Therefore, for a single harmonic number s, the beam–field coupling coefficient becomes

$$H_c = \sum_{n=-\infty}^{\infty} (A_{nm})^2 [J_s'(k_c r_l)]^2. \quad (82)$$

For a concentric beam, this means that if a gyrotron device operates at the sth gyration harmonic, the electron beam can have efficient interaction only with the field multipole of order $2s$. For other beam models, this statement is also valid if the field is expanded into the multipoles around the guiding centers of the gyrating electrons.

Since near the cyclotron resonance $\Omega_c \simeq \omega - k_z v_z / s$, $k_c = \omega / c \sin\psi$, $r_l = v \sin\phi / \Omega_c$, and $v_z = v \sin\phi$, we can write the argument of the Bessel function as [29]

$$k_c r_l \simeq \frac{s v \sin\phi \sin\psi}{c - v \cos\phi \cos\psi} \quad (83)$$

where ϕ is the pitch angle of the electron and ψ is the Brillouin angle of the wave mode, which is defined as $\tan^{-1}\psi = k_z / k_c$.

It is clear that if $v \sin\psi \sin\phi$ is very small compared to c, the speed of light in free space, i.e., if the electron beam is nonrelativistic or moves with a small pitch angle or both, then the Bessel function has an argument much smaller than its order s. This is because the higher the order of the Bessel function, the bigger the argument it needs to reach the first peak of the function value; in addition, the peak value of the Bessel function becomes smaller when its order increases. Therefore, when the harmonic number s increases, the coupling between the field and the electron beam becomes weaker. It is crucial to have a bigger Larmor radius for higher harmonic operation. This explains why, in general, the gyrotrons operating at higher gyration harmonics demand high electron beam energy and big $\alpha = u_t / u_z$ to have a larger portion of the electron kinetic energy in the transverse direction.

The waveguide structure is also critical for gyrotrons working efficiently at higher harmonics of the electron gyrofrequency because a good waveguide structure can achieve a much bigger component of the desired multipole field at the order of the harmonic number and, consequently, a much bigger beam–field coupling.

The inverse transformation of $\bar{F}(K)$ gives function $F(z)$, which is the function of frequency, the electron beam parameters, and the waveguide parameter.

The Laplace inverse transformation is defined as

$$F(z) = \int_{-jC-\infty}^{-jC+\infty} dK e^{jKz} \bar{F}(K) \quad (84)$$

where the contour C must be large enough to include all the poles of $\bar{F}(K)$.

The inverse transformation integral may be carried out by using the residue theorem in complex variable theory, i.e.,

$$F(z) = j \sum_{i=1}^{m} \mathrm{Res}[e^{jK_i z} \bar{F}(K_i)] \quad (85)$$

where K_i is the ith pole of the integrand. From (75), the poles of function $F(K)$ are the roots of its denominator $D(K)$ in (76).

In the "cold" electron beam case, $D(K)$ in (76) is a quartic function, and it is readily seen that all the singularities in $\bar{F}(K)$ are order of one. Because $\bar{F}(K)$ is in the form of (75), so we can write $F(z)$ in the following form by the residue theorem:

$$F(z) = j \sum_{i=1}^{4} e^{jK_i z} \frac{N(K_i)}{D'(K_i)} \quad (86)$$

where $D'(K)$ is the derivative of $D(K)$ with respect to K and is given by

$$D'(K) = -2\left[K\Omega_s^2(K) + \frac{u_z}{\gamma}\Omega_s(K)\left(\frac{\omega^2}{c^2} - K^2 - k_c^2\right)\right] - \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{-\infty}^{\infty} \frac{\pi v A_{nm}^2}{\gamma N} D_{10}'(K) \quad (87)$$

where

$$D_{10}'(K) = \frac{2K\beta_t^2}{c^2}[J_s'(k_c r_l) J_{m-s}(k_c R)]^2 + \frac{k_c r_l u_z}{\gamma}\left[2\Omega_e\left(1 - \frac{s^2}{k_c^2 r_l^2}\right) J_s(k_c r_l) J_{m-s}^2(k_c R) - \frac{k_c u_t}{2}\hat{\Psi}_-\right] J_s'(k_c r_l)$$
$$- \frac{2u_z}{\gamma}\Omega_s(K)\left[2\gamma k_c r_l\left(1 - \frac{s^2}{k_c^2 r_l^2}\right) J_s'(k_c r_l) J_{m-s}^2(k_c R) - \frac{k_c u_t}{\Omega_c}\hat{\Psi}_+\right] J_s(k_c r_l). \quad (88)$$

At the input plane $z=0$, if $F(0)$ is set to unity, we have a relation

$$j\sum_{l=1}^{4}\frac{N(K_l)}{D'(K_l)}=1. \tag{89}$$

Therefore, the problem of obtaining $F(z)$ is reduced to finding the poles of function $\bar{F}(K)$, or the roots in $D(K)$. Moreover, if the output of the gyro-TWA device is well matched to the load, the wave propagating in the negative z direction vanishes, and the corresponding coefficients in (86) and (89) must be zero.

Once $F(z)$ is determined, all the components of the field in any cross section in the waveguide can be obtained through (8) and (9). Since the field components in the waveguide are assumed to be in the forms in (2), (8), and (9), and $F(z)$ is given by (85) the power flow in the waveguide can be obtained by integrating the axial component of the Poynting vector over the cross-sectional area of the waveguide

$$\begin{aligned}P(z)&=\frac{1}{2}\operatorname{Re}\int_A dA\,\boldsymbol{e}_z\cdot(\boldsymbol{E}_t\times\boldsymbol{B}_t)\\&=\frac{1}{2}\operatorname{Re}\left\{-jF(z)\frac{\partial F(z)}{\partial z}\right.\\&\quad\left.\cdot\left[\frac{\omega}{ck_c^4}\int_A dA\nabla_t B_z^0\cdot(\nabla_t B_z^0)^*\right]\right\}.\end{aligned} \tag{90}$$

For a uniform waveguide, the part in the brackets in the above equation is independent of z. If we denote

$$D_t=-\frac{\omega}{2ck_c^4}\int_A dA\nabla B_z^0\cdot(\nabla B_z^0)^* \tag{91}$$

then we can write the power flow at z

$$\begin{aligned}P(z)&=D_t\operatorname{Re}\left[jF(z)\frac{dF(z)}{dz}\right]\\&=D_t\left\{\sum_{l=1}^{m}\left[e^{jK_lz}\frac{N(K_l)}{D'(K_l)}\right]\sum_{p=1}^{m}\left[K_m e^{jK_pz}\frac{N(K_p)}{D'(K_p)}\right]^*\right\}.\end{aligned} \tag{92}$$

If the beam–field interaction length is L, the gain in decibels is defined as

$$\begin{aligned}G&=10\log\frac{P(L)}{P(0)}\\&=10\log\left\{\frac{\left[\sum_{l=1}^{m}e^{jK_lL}\frac{N(K_l)}{D'(K_l)}\right]\left[\sum_{p=1}^{m}K_pe^{jK_pL}\frac{N(K_p)}{D'(K_p)}\right]^*}{\left[\sum_{i=1}^{m}\frac{N(K_i)}{D'(K_i)}\right]\left[\sum_{p=1}^{m}K_p\frac{N(K_p)}{D'(K_p)}\right]^*}\right\}\end{aligned} \tag{93}$$

where $P(L)$ is the power flow at the output end of the waveguide and $P(0)$ is the power flow at the input plane of the waveguide.

Obviously, the gain is a function of all the beam and waveguide parameters, and is also a function of the frequency. Thus, we can compute the gain as a function of the frequency with all the specified electron beam and waveguide parameters.

Due to the fact that the cyclotron maser radiation is strong only when the frequency is close to the cutoff frequency of the operating waveguide mode, for gyro-TWA devices with uniform waveguide, a relatively small bandwidth is expected; usually, the 3-dB bandwidth is just 1–3 percent. But, since the beam–field interaction is distributed along the waveguide, various methods can be used to alleviate this and a much bigger bandwidth can be achieved. Y. Y. Lau proposed a tapered instead of a uniform waveguide as the beam–field interaction space. Simultaneously, the applied magnetic field is also tapered to maintain the synchronization between the beam and the field along the waveguide. From both theoretical and experimental investigations, the gyro-TWA device with tapered waveguide can achieve about a 15-percent bandwidth centered at 35-GHz frequency [27]. Another method to increase the bandwidth is to decrease the applied magnetic field slightly below the grazing line, as many investigators have pointed out and confirmed in experiments [24], [27].

Analyzing the instability through the derivation of a dispersion equation is a generally used approach in plasma physics. Here, we can easily obtain a general dispersion equation from (76) for the gyro-TWA. The third term on the right-hand side of (78) is proportional to the square of $\Omega_s(K)$ and is much smaller compared to the other two terms near cyclotron resonance; this makes the neglect of that term on the right-hand side of (78) permissible. Therefore, if $D(K)$ in (76) is set to zero and the electron beam inhomogeneity is neglected, i.e., only the first two terms in the expression for $D_{10}(K)$ in (78) are taken, then a generalized dispersion equation for gyro-TWA can be obtained which is applicable to waveguides with any shape of cross section.

The zeroth-order solution to (6) may be assumed to be of the form e^{jk_zz} and k_z is understood as real, K is changed into k_z and frequency ω is assumed to be complex, as the common approach in plasma physics. Since $s\Omega_c\simeq\omega-k_zu_z/\gamma$, the generalized dispersion equation takes the form

$$\left(\frac{\omega^2}{c^2}-k_z^2-k_c^2\right)=\frac{\pi\nu}{\gamma N}\left[\frac{\beta_t^2(\omega^2-k_z^2c^2)}{\Omega_s^2}H-\frac{(\omega-k_zv_{z0})}{\Omega_s}Q'\right] \tag{94}$$

where

$$H=\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}[A_{nm}]^2[J_{m-s}(k_cR)J_s'(k_cr_l)]^2 \tag{95}$$

$$Q'=2H-\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}[A_{nm}]^2\left\{2k_cr_lJ_{m-s}^2(k_cR)\cdot J_s'(k_cr_l)J_s''(k_cr_l)-\frac{k_c^2r_l^2}{2}\Psi_+\right\}. \tag{96}$$

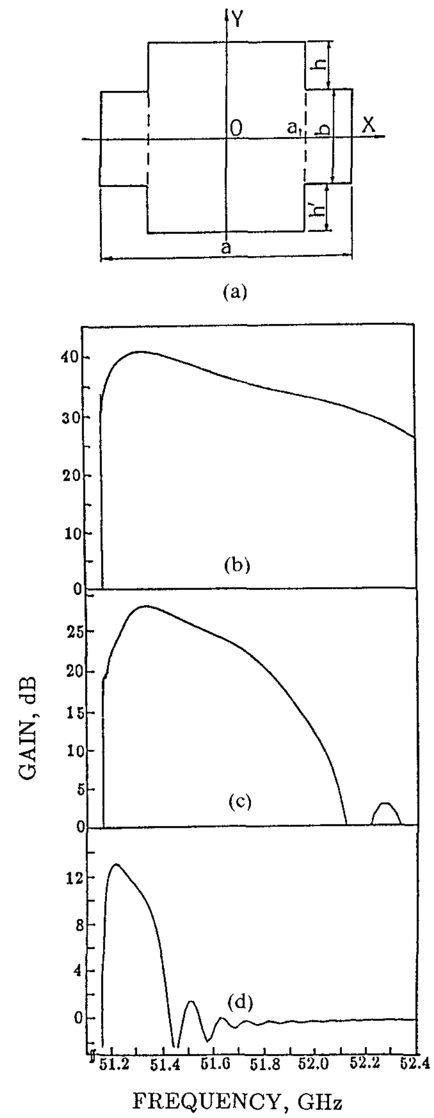

Fig. 3. (a) Cross section of an out-ridged waveguide: $a = 0.73$ cm, $b = 0.22$ cm, $a_1 = 0.285$ cm, $h = h' = 0.19$ cm. (b)–(d) Gain-frequency relation of gyro-TWA with out-ridged waveguide, TE_{02} mode: (b) $s = 2$, $\alpha = 2.0$, $V = 30$ kV, $I = 5$ A, $B_0 = 9.58$ kG, $L = 30$ cm; (c) $s = 4$, $\alpha = 3.0$, $V = 40$ kV, $I = 5$ A, $B_0 = 4.894$ kG, $L = 40$ cm; (d) $s = 6$, $\alpha = 5.5$, $V = 40$ kV, $I = 5$ A, $B_0 = 3.278$ kG, $L = 40$ cm.

Putting the coefficients A_{nm} to unity and removing the summations in (95) and (96), we obtain a dispersion equation suitable for the TE_{nm} mode of the circular waveguide if k_c is set to p_{nm}/a. Comparing this to the dispersion equation derived for the TE_{nm} mode of the circular waveguide in [24], we find that H is the same as H_{sm} in [24]. As to the function Q' in (96) and Q_{sm} in [24], the difference is between Ψ_+ in Q' and the last two terms in Q_{sm}. In Q', the Bessel functions $J_{s\pm1}(k_c r_l)$, $J_{s\pm2}(k_c r_l)$ and $J_{m-(s\pm1)}(k_c R)$, $J_{m-(s\pm2)}(k_c R)$ are also involved. In [30] and [31], Döhler explained that these may be important for "periotron" interaction.

IV. Computation of Gain–Frequency Function

To compute the gain–frequency function of a gyro-TWA with a given axial uniform waveguide structure, we need to specify all the parameters of the waveguide geometry and

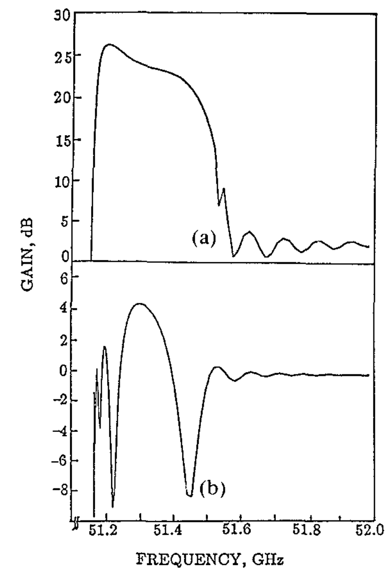

Fig. 4. Gain–frequency function of gyro-TWA with out-ridged waveguide: $a = 2.2$ cm, $b = 0.662$ cm, $a_1 = 0.857$ cm, $h = h' = 0.57$ cm, TE_{62} mode. (a) $s = 2$, $\alpha = 2.0$, $V = 30$ kV, $I = 5$ A, $B_0 = 9{,}58$ kG, $L = 30$ cm; (b) $s = 4$, $\alpha = 3.0$, $V = 40$ kV, $I = 5$ A, $B_0 = 4.894$ kG, $L = 40$ cm.

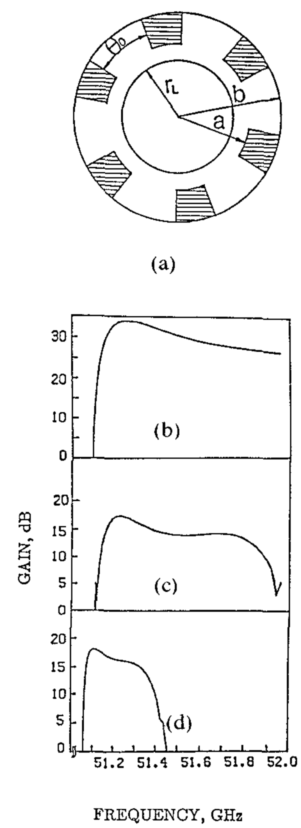

Fig. 5. (a) Cross section of a magnetron-type waveguide: (b)–(d) Gain–frequency relation: (b) $s = 2$, $\alpha = 0.187$ cm, $b = 0.281$ cm, $N_d = 2$, $I = 5$ A, $V = 30$ kV, $\alpha = 2.0$, $\theta_0 = 60°$, $k_c = 10.7$, $B_0 = 9.575$ kG, $L = 30$ cm; (c) $s = 4$, $a = 0.221$ cm, $b = 0.335$ cm, $N_d = 4$, $I = 5$ A, $V = 40$ kV, $\alpha = 3.0$, $\theta_0 = 40°$, $k_c = 10.7$, $B_0 = 4.896$ kG, $L = 40$ cm; (d) $s = 6$, $a = 0.228$ cm, $b = 0.342$ cm, $N_d = 6$, $I = 5$ A, $V = 40$ kV, $\alpha = 2.0$, $\theta_0 = 30°$, $k_c = 10.7$, $B_0 = 3.276$ kG, $L = 40$ cm.

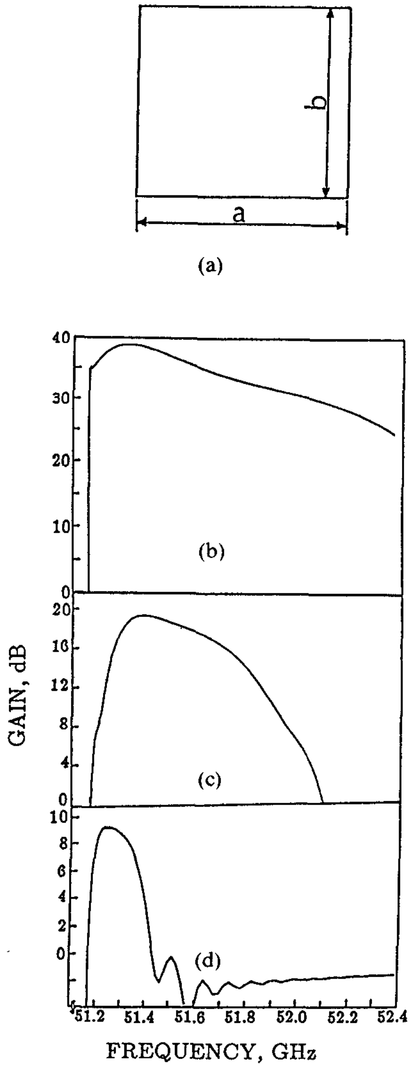

Fig. 6. (a) Cross section of a rectangular waveguide: $a = 0.586$ cm, $b = 0.6$ cm: (b)–(d) Gain–frequency relation: (b) $s = 2$, $\alpha = 2.0$, $V = 30$ kV, $I = 5$ A, $B_0 = 9.58$ kG, $L = 30$ cm, TE_{02}; (c) $s = 4$, $\alpha = 3.0$, $V = 40$ kV, $I = 5$ A, $B_0 = 4.894$ kG, $L = 40$ cm, TE_{02}; (d) $s = 6$, $\alpha = 5.5$, $V = 40$ kV, $I = 5$ A, $B_0 = 3.278$ kG, $L = 40$ cm, TE_{02}.

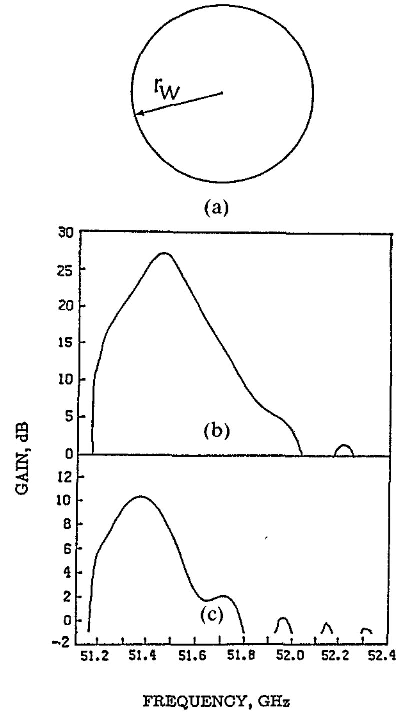

Fig. 7. (a) Cross section of a circular waveguide: (b)–(c) Gain–frequency relation: (b) $s = 2$, $\alpha = 2.0$, $V = 30$ kV, $I = 5$ A, $B_0 = 9.58$ kG, $L = 30$ cm, $r_w = 0.562$ cm, TE_{22} mode; (c) $s = 4$, $\alpha = 3.0$, $V = 40$ kV, $I = 5$ A, $B_0 = 4.894$ kG, $L = 40$ cm, $r_w = 0.496$ cm, TE_{41} mode.

those of the electron beam; we also need to obtain the coefficients in the series of the expansion of the axial magnetic field in the waveguide and the norm of the axial magnetic field given by (7). Then we can use the results of the kinetic theory to compute the gain–frequency curves.

The details for calculating the field coefficients and the norm of the axial magnetic field are given in [32]. Some computed results of the gain–frequency function of the gyro-TWA's with out-ridged, magnetron-type, rectangular, and circular waveguides are shown in Figs. 3–7 along with the waveguide and electron beam parameters. For convenience of comparison, we set the same beam parameters and the same length of interaction for the gyro-TWA's with different waveguides. From the plotted results, it is seen that the gyro-TWA's with the widely used simple circular waveguide and rectangular waveguide have high gain at lower cyclotron harmonics; the gain decreases rapidly as the harmonic number becomes higher. However, the gyro-TWA's with the out-ridged and with the magnetron-type waveguide still have relatively high gain at higher cyclotron harmonics.

V. Conclusions

A unified single-mode theory has been developed for gyro-TWA at cyclotron harmonics, both in the nonlinear and the linear regimes. The waveguide fields are expanded into the series of multipoles around the guiding centers. The theory is applicable to a wide class of waveguide cross sections and waveguide modes with arbitrary harmonic numbers and with the generalized electron beam model. Some common features of the gyro-TWA operating at the higher harmonics are explored. The general dispersion equation is derived. Some numerical examples of the gain–frequency curves for gyro-TWA's with several different waveguide structures are provided for comparison.

Acknowledgment

The authors would like to thank Prof. I. B. Bernstein, Prof. A. Hezenberg, Prof. R. Jensen, and Prof. K. R. Chu for stimulating discussions.

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